Physics 341 – Advanced Physics Laboratory

The Michelson Interferometer and the Hydrogen-Deuterium Isotope Shift

I. Introduction

The Michelson interferometer, illustrated below, is a simple instrument capable of highly precise measurements. Invented in 1880 by Albert Michelson and used shortly thereafter in his famous experiment to determine the speed and direction of the aether wind, Michelson and others soon realized the instrument's great versatility, and the interferometer was soon put to use in such diverse applications as measuring the wavelength and spectral width of atomic emission lines, defining the meter in terms of the wavelength of light, measuring the diameters of stars (via the spatial coherence of starlight), and measuring the thickness and surface features of microscopically thin films. With the invention of the laser in 1960, the field of laser interferometry (including the Michelson interferometer as just one configuration) became a major branch of optical spectroscopy and an important experimental tool for investigating atomic/molecular and condensed matter systems.

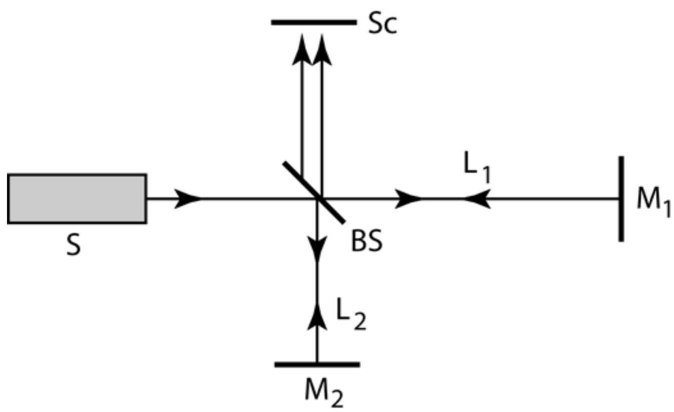

Fig. 1. Simplified Michelson interferometer. S = light source, BS = beamsplitter, M_1 and M_2 = mirrors, Sc = viewing screen.

The operating principle of the Michelson interferometer is easily explained. Light from a source S is split by a beamsplitter BS (half-silvered mirror). One beam propagates to mirror M_1 and returns to the beamsplitter, the other propagates to mirror M_2 and returns. The superimposed output beams are viewed on screen Sc, where an interference pattern is visible. If the path lengths L_1 and L_2 differ by a quarter wavelength, then the round-trip propagation distance differs by a half wavelength, and the output beams interfere destructively. If a glass plate of varying thickness is placed in one of the arms, then the distance light travels varies slightly for different parts of the plate, giving rise to a pattern of bright and dark interference fringes corresponding to different thicknesses of the plate.

In this experiment, you will illuminate the interferometer with a hydrogen-deuterium lamp. The lamp contains an electrically excited gas which is a mixture of hydrogen and deuterium, so the interferometer is illuminated simultaneously by the electronic spectra from both atoms. A colored filter is used to isolate the $n = 3$ to $n = 2$ transition for both atoms. Since the transitions in hydrogen occur at slightly different wavelengths than in deuterium, the interference patterns of the two slightly different wavelengths are shifted spatially. By measuring the interference patterns, you will be able to measure the slight difference in wavelength between hydrogen and deuterium.

II. Prelab Experiment: Interferometric Measurement of the Refractive Index of Optical Glass

A. Description of the Experimental Apparatus

In this experiment, we will use the Michelson interferometer to measure the index of refraction of a glass slab. By rotating the slab through a small angle, its effective length can be slightly changed, leading to a measurable shift of the interference fringe pattern. By counting the fringe shift and measuring the rotation angle of the slab accurately, one can deduce the refractive index of the glass material. Although straightforward in principle, reckoning the fringe shift involves a nontrivial but instructive geometrical calculation to determine the change in optical path length. The refractive index of the slab will also be measured in another configuration, by observing the transverse deviation of a laser beam propagated through the slab.

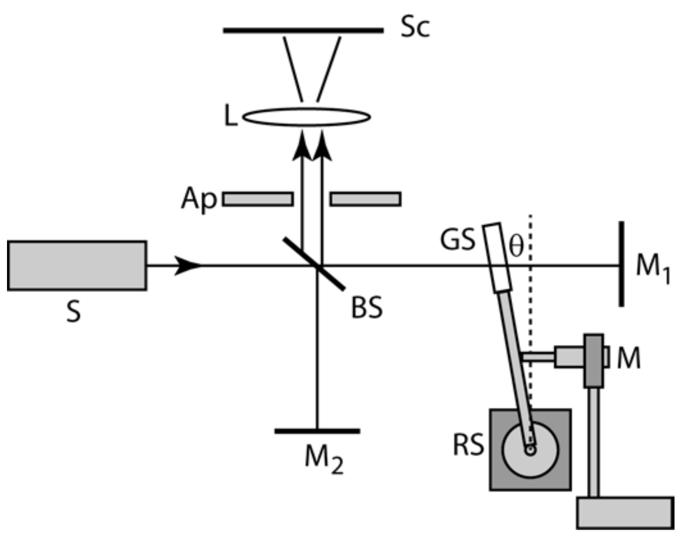

Figure 2. Michelson interferometer with glass slab (GS) in one arm. S = laser source, M_1 and M_2 = mirrors, BS = beamsplitter, M = micrometer screw held in right-angle clamp, RS = rotation stage, clamped to table, θ = angle of rotation from normal incidence, Ap = aperture, L = lens, Sc = screen for viewing interference pattern.

Experimental Procedure

1. Mount the glass slab in its holder so that it is in the light path of one arm of the interferometer. Check that the fringe pattern is visible on the screen, with one or two dark fringes visible. Small adjustments in the mirror tilts may help optimize the appearance of the interference pattern on the screen. If adjustments of the mirror tilts are ineffective, ask for help, since alignment of the interferometer may be needed.

Caution: Be careful not to touch the delicate first-surface coatings of the mirrors and beamsplitter.

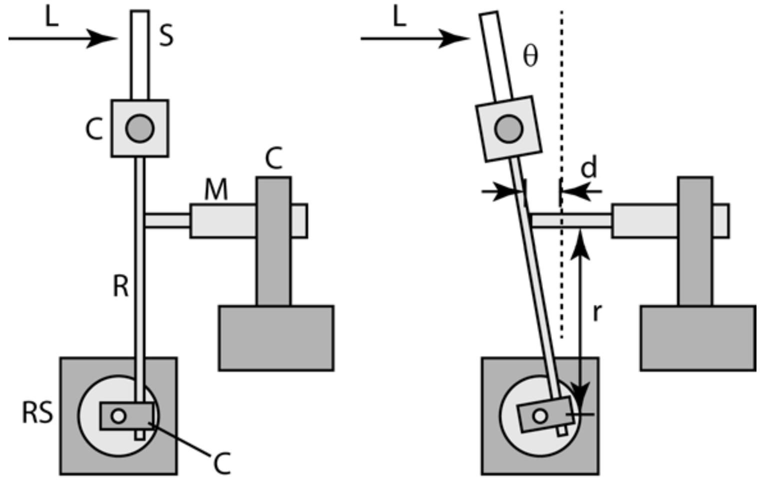

Figure 3. Diagram of the apparatus for rotating a glass slab by a small angle in the light path of the Michelson interferometer. S = glass slab, L = light path, C = clamp, M = micrometer screw held in a right-angle clamp, RS = rotation stage clamped to optical table, R = rod. In the picture on the right, the slab is rotated through angle θ given by tan θ = d/r.

2. Adjust the rotation angle of the slab for normal incidence (so that the reflected spots from the slab are reflected back to the laser). Adjust the micrometer M, held in a clamp, to push on the rod R holding the slab, as shown in the diagram above. Record the starting micrometer reading when the slab is normal to the incident light. Position the screen with a reference mark aligned along a dark fringe.

3. Slowly advance the micrometer screw so that it pushes rod R, rotating the slab through a small angle θ. Count the number N of fringes that pass. When you reach $N = 20$ (or so) fringes, stop and record the micrometer reading. Carefully measure the distance r from the axis of rotation to the point of contact of the micrometer screw with the rod. The angle θ is given by $\tan \theta = d/r$, where d is the distance the micrometer screw advanced. Knowing both N and θ, one can determine the index of refraction n of the glass slab. A derivation is outlined in the next section.

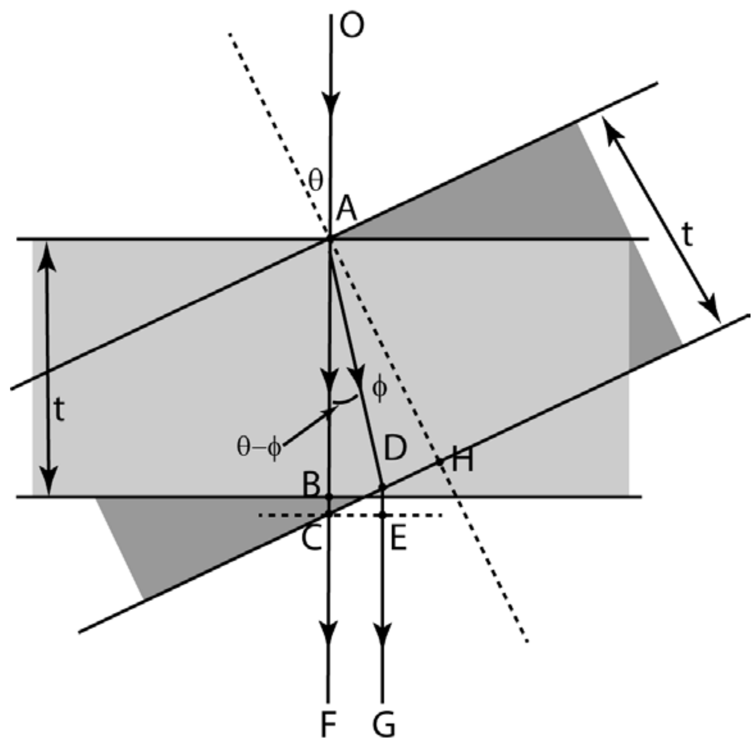

Figure 4. Geometry for calculating the number of fringes appearing as the glass slab is rotated.

B. Calculation of the Refractive Index

The geometry for the derivation is shown above. At normal incidence, the light propagates along the straight line through points *O*, *A*, *B*, *F*; when the slab is tilted by angle θ, the light propagates along the straight segments *OA*, *AD*, *EG*. The thickness of the slab is *t*. The number *N* of fringes that pass when the slab is tilted is given by

$$2\,\Delta\mathrm{OPL} = \lambda\, N \tag{1}$$

where ΔOPL is the change in the optical path length (OPL) of the slab. Optical path length is defined as the product of distance × refractive index, summed over all parts of the light path. The OPL before rotation is given by

$$\mathrm{OPL}_i = n\,AB + BC, \tag{2}$$

where *n* is the refractive index of the slab, and *AB* denotes the length of the segment joining points *A* and *B*. Likewise, the OPL after rotation is given by

$$\mathrm{OPL}_f = n\,AD + DE. \tag{3}$$

Now we can express the distances *AB*, *BC*, *AD*, and *DE* in terms of the thickness *t* of the slab, the incident angle θ, and the refracted angle ϕ using geometry. First, note that

$$AD = t\ \sec\phi, \tag{4}$$

$$AB = t, \tag{5}$$

$$DE = CE\ \tan\theta. \tag{6}$$

By noting the angle $\angle BAD = \theta - \phi$, it can easily be shown that

$$CE = AD\ \sin(\theta-\phi) = t\ \sec\phi\ \sin(\theta-\phi). \tag{7}$$

Then combining Eqs. (6) and (7), we obtain

$$DE = t\ \sec\phi\ \sin(\theta-\phi)\ \tan\theta. \tag{8}$$

Considering the triangle $\triangle\, ACH$, it follows from geometry that

$$BC = t \sec\theta - t. \qquad (9)$$

Then, combining Eqs. (1), (2), and (3) along with the subsequent relations yields

$$\lambda \frac{N}{2} = n\,t \sec\phi + t \sec\phi \sin(\theta - \phi)\tan\theta - n\,t - t \sec\theta + t. \qquad (10)$$

From trigonometry, it follows that

$$\sec\phi \tan\theta \sin(\theta - \phi) = (\tan\theta - \tan\phi)\sin\theta \qquad (11)$$

Combining the previous two results and simplifying then yields

$$\lambda \frac{N}{2} = \frac{t}{\cos\phi}(n - \sin\theta\sin\phi) + t\,(1 - n - \cos\theta). \qquad (12)$$

Snell's law, $\sin\theta = n \sin\phi$, allows us to eliminate the refracted angle ϕ in favor of the incident angle θ, yielding

$$\frac{\lambda N}{2} = t\left(\frac{n - \frac{1}{n}\sin^2\theta}{\sqrt{1 - \frac{1}{n^2}\sin^2\theta}}\right) + t\,(1 - n - \cos\theta). \qquad (13)$$

which subsequently can be simplified to

$$\frac{\lambda N}{2} = t\sqrt{n^2 - \sin^2\theta} + t\,(1 - n - \cos\theta). \qquad (14)$$

Isolating the radical in the above expression and squaring both sides and solving for *n* finally yields the desired relation

$$n = \frac{\left(t - \frac{1}{2}\lambda N\right)(1 - \cos\theta)}{t\,(1 - \cos\theta) - \frac{1}{2}\lambda N}. \qquad (15)$$

In deriving the above, a term of order $\left(\frac{\lambda N}{2}\right)^2$ was dropped since it is much smaller than other terms adding to it. Eq. (15) allows one to calculate the refractive index *n* of the glass slab, knowing its thickness *t*, the wavelength λ, and the number *N* of fringes that pass when the slab is rotated by angle θ. A cautionary note on evaluating Eq. (15) numerically: Since in an experiment, θ may be a very small angle so that the term $1 - \cos\theta$ involves the difference of quantities both nearly equal to unity, hence it is important to take care to keep enough extra digits that accuracy is not compromised (of course, the final result should always be reported with an appropriate number of significant digits).

C. Prelab Assignment

Carry out the above-described experiment to determine the refractive index of the glass slab. Then answer the following questions.

1. The first question involves working through the derivation presented in Section II B. Before starting, you should carefully read through the derivation up to Eq. 7 and be sure you follow the reasoning. To "derive" the remaining steps, present the necessary intermediate steps, making reference to the diagram or any needed auxilliary diagrams you draw.

a. Derive Eq. 7.
b. Derive Eq. 9.
c. Derive Eq. 11.
d. Derive Eq. 12.
e. Derive Eq. 13.
f. Derive Eq. 14.
g. Derive Eq. 15.

2. Using your experimental data and Eq. 15, find the refractive index of the glass slab. Show all calculations.

III. Nuclear Mass and the Hydrogen-Deuterium Isotope Shift

In 1913, Niels Bohr solved a semiclassical model and was able to account for the energy levels the hydrogen atom for the first time, obtaining the famous result $E_n = -\frac{E_0}{n^2}$, where $E_0 = \frac{m_e(k\,e^2)^2}{2\hbar^2} = 13.61\ eV$ and *n* is a positive integer, with *k* the Coulomb's law constant and m_e the electron mass. A more rigorous approach to obtain the Bohr energies proceeds by solving the Schrodinger equation for the Coulomb potential. If we model the nucleus as immovable and fixed at the origin, we obtain the same energies found by Bohr. In reality, both the electron and the nucleus are moving and the Schrodinger equation for this two-body problem can be solved exactly. The following discussion outlines the steps in the solution of the two-body problem. We will find that the consideration of the finite nuclear mass leads to a small correction of the energy levels of the Bohr model, so that different isotopes of an element are predicted to have slightly different spectra.

We start by writing the Schrodinger equation for a two-particle system moving in a central potential, given by

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V(|\vec{r}_1 - \vec{r}_2|)\psi = E\psi,$$

where the wavefunction $\psi = \psi(\vec{r}_1, \vec{r}_2)$ is a function of the coordinates $\vec{r}_1$ and $\vec{r}_2$ of the two particles and the potential *V* depends only on the distance between the particles. Clearly the hydrogen atom is a specific example of the above form with $V(|\vec{r}_1 - \vec{r}_2|) = -\frac{k\,e^2}{|\vec{r}_1 - \vec{r}_2|}$. The notation ∇_1^2 represents the derivatives with respect to $\vec{r}_1 = (x_1, y_1, z_1)$; that is, $\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2}$. We now change to new variables $\vec{r}$ and $\vec{R}$ defined by $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$. Note that if $\vec{r}_1$ represents the electron and $\vec{r}_2$ the nucleus, $\vec{r}$ is the position of the electron relative to the nucleus and $\vec{R}$ is the center-of-mass vector. Solving for $\vec{r}_1$ and $\vec{r}_2$, we find $\vec{r}_1 = \vec{R} + \frac{m_2\,\vec{r}}{m_1 + m_2}$ and $\vec{r}_2 = \vec{R} - \frac{m_1\,\vec{r}}{m_1 + m_2}$. Making use of the chain rule, we can show that $\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2 = \frac{1}{m_1 + m_2}\nabla_{cm}^2 + \frac{1}{\mu}\nabla_{rel}^2$ where ∇_{cm}^2 involves derivatives with respect to the coordinates of $\vec{R}$ and ∇_{rel}^2 involves derivatives with respect to the coordinates of $\vec{r}$. The parameter μ, called the reduced mass, is given by $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$. In terms of the new variables, we can thus write the Schrodinger equation as

$$-\frac{\hbar^2}{2(m_1 + m_2)}\nabla_{cm}^2\psi - \frac{\hbar^2}{2\mu}\nabla_{rel}^2\psi + V(r)\psi = E\psi.$$

The solution proceeds by the familiar technique of separation of variables; as usual, we take ψ to be a product of wavefunctions as $\psi = \psi_{cm}(\vec{R})\psi_{rel}(\vec{r})$. Plugging in and dividing both sides by ψ, we find

$$\left(-\frac{\hbar^2}{2(m_1 + m_2)}\frac{\nabla_{cm}^2\psi_{cm}}{\psi_{cm}}\right) + \left(-\frac{\hbar^2}{2\mu}\frac{\nabla_{rel}^2\psi_{rel}}{\psi_{rel}} + V(r)\right) = E.$$

Since the first bracketed term is a function of $\vec{R}$ only and the second is a function of $\vec{r}$ only, yet the sum is constant (E), we can conclude that each of the bracketed terms is separately constant. Thus we can write

$$-\frac{\hbar^2}{2(m_1+m_2)}\nabla^2_{cm}\psi_{cm} = E_1\psi_{cm} \text{ and}$$

$$-\frac{\hbar^2}{2\mu}\nabla^2_{rel}\psi_{rel} + V(r)\psi_{rel} = E_2\psi_{rel},$$

with $E_1 + E_2 = E$. Let's assume the center-of-mass of the sample is stationary in the lab. The second equation then gives the energy spectrum of the hydrogen atom. The second equation is the Schrodinger equation of a (fictitious) particle of mass μ moving a fixed potential and the solutions are identical to those found by Bohr, but with m_e replaced by μ.

IV. Experimental Measurement of the H-D Isotope Shift

A. Circular Fringe Pattern

Light from a spatially extended source produces a set of circular fringes resembling a bullseye when the output of the interferometer is viewed with the eye or a telescope. To understand how circular fringes are produced, consider the diagram in Figure 5. Light is emitted by an extended source S and, after passing the beamsplitter, is incident on mirrors M_1 and M_2. In the figure, the two perpendicular arms of the interferometer have been depicted as parrallel and superimposed and the beamsplitter omitted. Plane mirror M_1 forms an image S_1 of the source and mirror M_2 forms image S_2. Since the mirrors are separated by distance *d*, the images are separated by 2*d* as shown. The dotted line in the figure indicates the axis of the interferometer, perpendicular to the mirrors. A light ray making angle θ with repsect to the axis is reflected at angle θ (law of reflection). Parallel rays R_1 and R_2 emitted at angle θ interfere constructively when their path difference ($2d\cos\theta$) is an integer number of wavelengths, i.e. $2d\cos\theta = m\lambda$ where *m* is an integer. For an extended source, each point on the source emits rays in all directions, and different points on the source are not synchronized. Nevertheless, when $\frac{2d\cos\theta}{\lambda}$ is an integer, rays emitted by each point on the source that are split and brought together will interfere constructively, so all source points contribute to a bright fringe at angle θ. The system is symmetric about its axis, so the rays at a given angle θ form a cone about the axis of symmetry, resulting in a circular fringe pattern.

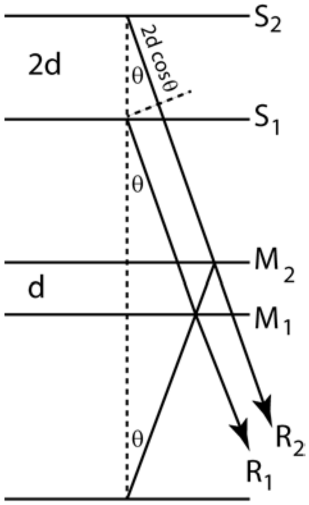

Fig. 5

B. Measuring Small Wavelength Differences

When the source emits two different wavelengths, two different bullseye interference patterns are superposed. Consider the center of the bullseye pattern at $\theta = 0$. Light of wavelength λ_1 produces a bright spot when the condition $2d = m_1\lambda_1$ is satisfied with m_1 an integer; likewise, light of wavelength λ_2 interferes constructively when $2d = m_2\lambda_2$. Now suppose the path difference *d* is slowly varied by translating one of the mirrors. Both bullseye pattens vary, with new fringes appearing in the center and expanding outward as *d* is increased. The fringe patterns coincide when $m_1 = m_2 + N$, with N an integer. When the fringe patterns coincide, bright and dark fringes of high contrast are visible. However when the patterns do not coincide, the superposition of the patterns results in inexact alignment of the bright fringes of each wavelength, causing the appearance of a blurred interference pattern. When the bright fringes from wavelength λ_1 align with the dark fringes from wavelength λ_2, the pattern disappears (looks uniformly illuminated)—remember that λ_1 and λ_2 are nearly the same. The next coincidence, after translating the mirror by distance Δd, occurs when $m_1 = m_2 + N + 1$. Writing $m_1 = 2d/\lambda_1$ and $m_2 = 2d/\lambda_2$, we can express the conditions for the two successive coincidences as

$$\frac{2d}{\lambda_1} = \frac{2d}{\lambda_2} + N \text{ and } \frac{2(d+\Delta d)}{\lambda_1} = \frac{2(d+\Delta d)}{\lambda_2} + N + 1.$$

Subtracting the first relation from the second yields $\frac{2\Delta d}{\lambda_1} = \frac{2\Delta d}{\lambda_2} + 1$, which can be rewritten as

$$\frac{\lambda_2 - \lambda_1}{\lambda_1 \lambda_2} = \frac{1}{2\Delta d}.$$

Recalling that $\lambda_1 \approx \lambda_2$ and denoting $\lambda_2 - \lambda_1 = \Delta\lambda$ yields

$$\Delta\lambda = \frac{\lambda^2}{2\Delta d}.$$

The above formula allows one to calculate the small difference in wavelength $\Delta\lambda$ by measuring the distance Δd between successive coincidences of the interference pattern.

C. Experimental Set-Up

The experimental set-up for measuring the wavelength difference between the Hα line for hydrogen and deuterium is shown in Fig. 6 below. The Hα line corresponds to the $n = 3$ to $n = 2$ transition and has an approximate wavelength of 656 nm.

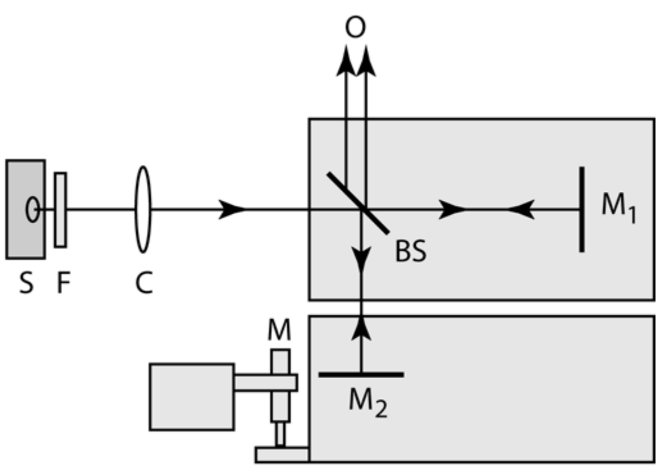

Fig. 6

A lamp containing a mixture of hydrogen and deuterium gas excited by an electric discharge is used as the light source (Lambda Scientific Model LLE-8). A red filter F placed just behind the lamp aperture is used to isolate the Hα line. A collimating lens L is used to capture the diverging cone of rays from the lamp and render them parallel (collimated). To accomplish this, the lens should be positiioned so that the source is at its focal point. You may need to measure the focal length of the collimating lens in a separate experiment (e.g. imaging winow light on a screen). the source, lens, and interferometer should be positioned at the same height. Collimated light is incident on the beamsplitter BS of the interferometer as usual, and light propagates along the two arms before being recombined at the beamsplitter and viewed by observer O. One of the mirrors can be translated using micrometer screw M. Be very careful not to turn the micrometer screw past its allowed travel range. To measure the coincidences in the interference pattern, decide on the direction the mirror will be translated and set the micrometer at the appropriate starting position. Then turn the micrometer slightly in the direction it will move in order to eliminate backlash in the micrometer screw mechanism, and then slowly advance the screw unidirectionally while observing the interference pattern. You can measure the micrometer positions at the coincidences (best focus) or anti-coincidences (where the interference pattern is unobservable). For best accuracy, advance the micrometer screw unidirectionally, or if the screw is reversed, back up enough to pass the backlash and re-advance unidirectionally to the final micrometer position; that is, all final screw positions should be recorded with the micrometer screw traveling in the same direction. Several trials may be taken and averaged for greater accuracy.

V. References

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Tipler, Paul A. Modern Physics. (Worth, New York, 1978). Any book on modern physics will contain a discussion of the Bohr model and many discuss reduced mass and the two-body problem.